Analytical and Numerical Modeling of Astrophysical X-Ray and Gamma-Ray Transients

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Abstract

The research performed under the above titled ONR grant has resulted in a number of important advances in the treatment of integrodifferential transport equations, such as those obtained when the Fokker-Planck coefficients depend on the "temperature" of the particle distribution, as defined by a suitable integral. In summary, we have developed a new technique for independently determining the temperature as a function of time using the energy moments of the initial distribution. This is equivalent to decoupling the original integrodifferential problem into two parts: (i) determination of the temperature variation, and (ii) determination of the particle distribution as a function of time and energy. **This represents a major advance in the treatment of integrodifferential Fokker-Planck equations because the two problems obtained can be solved separately without recourse to traditional predictor-corrector alogorithms.** The technique has been applied with success to the problem of self-consistent Comptonization in astrophysical plasmas. The theoretical approach utilizes a novel combination of physics, applied mathematics, and advanced symbolic computational techniques, and shows good promise for generalization to the treatment of other integrodifferential systems of interest to the Navy. The multi-disciplinary CSI program at George Mason University has proven an ideal venue for this investigation.

I. Introduction

During radiation-dominated X-ray transients in AGNs, energy is rapidly transferred between the photons and the electrons until the electron temperature (T_e) has equilibrated to the inverse-Compton temperature of the radiation ($T_{\rm IC}$). This occurs on a much shorter timescale than the equilibration of the radiation into a Wien distribution. Once the temperatures have equilibrated, the condition $T_e = T_{\rm IC}$ continues to be enforced by the Compton exchange of energy between the photons and the electrons. The radiative transfer problem then becomes highly nonlinear, since the variation of the electron temperature determines the spectrum (through the Kompaneets equation), while the spectrum determines the electron temperature (through the condition $T_e = T_{\rm IC}$). It is therefore extremely useful to develop techniques for independently calculating the electron temperature as a function of time for an arbitrary initial radiation spectrum.

In the research carried forth here, we have developed a general technique for determining the electron temperature by using the energy moments of the initial spectrum to construct a continued fraction with dramatic convergence properties. The electron temperature so obtained is then folded back into the PDE to compute the spectral solution. Hence the original integrodifferential problem is decoupled into two parts: (i) determination of the electron temperature as a function of time, and (ii) determination of the photon spectrum as a function of time and energy.

II. Time-Dependent Comptonization

In radiation-dominated, fully-ionized plasmas, photon creation and destruction are unable to establish local thermodynamic equilibrium, and the photons and electrons interact primarily via Compton scattering. Neglecting thermal opacity sources, the temporal evolution of a distribution of photons in an infinite, homogeneous plasma with electron number density n_e and electron temperature T_e is governed by the Kompaneets (1957) Fokker-Planck equation

$$\frac{\partial \bar{n}}{\partial t} = \frac{n_e \sigma_{\rm T} h}{m_e c} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(\bar{n} + \frac{kT_e}{h} \frac{\partial \bar{n}}{\partial \nu} \right) \right] , \tag{1}$$

where $\bar{n}(\nu, t)$ is the photon occupation number, ν is the photon frequency, and the terms on

the right-hand side express the effects of recoil losses and energy diffusion, respectively. Note that the properties of the electrons need not be constant, and as we argue below, the electron temperature will track the inverse-Compton temperature of the photons during radiation-dominated flares.

A Maxwellian distribution of electrons will exchange no net energy with the photons if $T_e = T_{\rm IC}$, where

$$T_{\rm IC} \equiv \frac{1}{4k} \frac{\int_0^\infty h\nu^4 \, \bar{n} \, d\nu}{\int_0^\infty \nu^3 \, \bar{n} \, d\nu} \tag{2}$$

is the inverse-Compton temperature of the radiation, defined here without regard to stimulated processes. In a radiation-dominated plasma, energy is rapidly transferred between the photons and the electrons until $T_e = T_{\rm IC}$. This occurs on a much shorter timescale than the equilibration of the radiation into a Wien distribution $\bar{n} \propto \exp(-h\nu/kT_e)$. If the cooling timescale is much shorter than the diffusion timescale ($t_d \gg t_c$), then most of the spectral evolution takes place *after* the electron and inverse-Compton temperatures have equilibrated. During this later phase, the condition $T_e = T_{\rm IC}$ is enforced by the Compton exchange of energy between the two species (Becker & Begelman 1986), and Compton scattering drives the photon distribution toward a Wien distribution.

The physical requirement that $T_e = T_{\rm IC}$ in the Kompaneets equation increases the complexity of the problem, since the equation becomes integrodifferential in nature. The temperature must therefore be computed self-consistently along with the spectrum, starting from a specified initial radiation distribution. The coupled integrodifferential system is usually solved numerically using predictor-corrector techniques. **However, we show here that the variation of the electron temperature can be determined *in advance* using only the initial spectrum itself.** With the electron temperature independently determined, the solution for the spectrum can be generated straightforwardly by numerical integration of the Kompaneets equation. In § III we develop a new technique for computing the perturbation series for the temperature using only the energy moments of the initial spectrum. In § IV the resulting (divergent) power series is transformed into a (convergent) continued fraction and evaluated for a variety of initial spectra.

III. Perturbation Method

The Kompaneets equation can be rewritten in terms of the dimensionless energy

$$x \equiv \frac{h\nu}{m_e c^2}, \tag{3}$$

and the Compton y-parameter (essentially the dimensionless time)

$$y \equiv \int_0^t \frac{kT_e(t')}{m_e c^2} n_e \sigma_{\mathrm{T}} c\, dt', \tag{4}$$

as

$$\frac{\partial \bar{n}}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\bar{n}}{\theta} + \frac{\partial \bar{n}}{\partial x}\right)\right], \tag{5}$$

where

$$\theta \equiv \frac{kTe(t)}{m_e c^2}. \tag{6}$$

Next we introduce the *energy moments* of the radiation spectrum, defined as

$$I_\ell(y) \equiv \int_0^\infty x^\ell \bar{n}\, dx, \tag{7}$$

and note that the photon energy and number densities can be expressed as

$$n_{\mathrm{ph}} = \frac{8\pi}{c^3}\int_0^\infty \nu^2 \bar{n}\, d\nu = 8\pi\left(\frac{m_e c}{h}\right)^3 I_2, \tag{8}$$

and

$$u_{\mathrm{ph}} = \frac{8\pi}{c^3}\int_0^\infty h\nu^3 \bar{n}\, d\nu = 8\pi m_e c^2\left(\frac{m_e c}{h}\right)^3 I_3. \tag{9}$$

We also note that the condition $T_e = T_{\mathrm{IC}}$ requires that

$$\theta = \frac{I_4}{4I_3}, \tag{10}$$

and operate on equation (5) with $\int_0^\infty x^\ell\, dx$ to obtain the *differential recurrence relation*

$$\frac{dI_\ell}{dy} = (\ell - 2)\left[(\ell+1)I_\ell - \frac{4I_3 I_{\ell+1}}{I_4}\right]. \tag{11}$$

Since there can be no net energy exchange between the photons and the electrons ($T_e = T_{\mathrm{IC}}$), it follows that I_3 =constant. Equations (10) and (11) can therefore be applied recursively to

obtain the y-derivatives of θ in terms of the moments of the radiation spectrum. In particular, the derivatives of θ evaluated at $y = 0$ can be expressed in terms of the moments of the initial radiation distribution, which are known. For example, the zeroth and first derivatives of θ at $y = 0$ are respectively

$$\theta_0 = \frac{I_4(0)}{4I_3(0)}, \tag{12}$$

$$\theta_0^{(1)} = \frac{I_3(0)}{2I_4(0)} \left\{ 5 \left[\frac{I_4(0)}{I_3(0)} \right]^2 - 4\frac{I_5(0)}{I_3(0)} \right\}, \tag{13}$$

etc. In Figure 1 we plot the Taylor series for $\theta(y)$ obtained in the case of a bremsstrahlung initial spectrum,

$$\bar{n}(x,y)|_{y=0} = x^{-3}\, e^{-x/4\theta_0}. \tag{14}$$

The series is obviously divergent beyond a small radius of convergence. However, we show in § IV that the series can be recast as a convergent continued fraction.

IV. Continued-Fraction Representation

Although the perturbation series obtained using the energy moments of the initial spectrum is often quite divergent, useful information about the temperature variation can nonetheless be recovered by using the power-series coefficients to construct the continued-fraction representation

$$\theta(y) = \cfrac{c_0}{1 + \cfrac{c_1\, y}{1 + \cfrac{c_2\, y}{1 + \ddots \cfrac{c_{N-1}\, y}{1 + c_N\, y}}}}, \tag{15}$$

which is simply a rational function approximation to the unknown function $\theta(y)$. The continued-fraction coefficients $c_0, \ldots, c_N$ are computed using the algorithm derived by Becker (1988). The resulting sequence of continued fractions is plotted in Figure 2 for the case of a bremsstrahlung initial spectrum. In contrast to the highly divergent Taylor series, the sequence of continued fractions is *uniformly convergent* to a unique *limit function*. The general form of the temperature variation is in good agreement with the numerical calculations and semi-analytic models of

Becker & Begelman (1986). Furthermore, the run of $\theta(y)$ computed by substituting the resulting solution for the photon spectrum $\bar{n}(x, y)$ into equations (7) and (10) agrees nearly perfectly with the limit function depicted in Figure 2.

We have also obtained results for monochromatic initial spectra and for combinations of power-laws and exponentials, and we find that the sequence of fractions is convergent in all cases, although uniform convergence is achieved only when the initial photon number spectrum diverges logarithmically at low energy. The observed X-ray spectrum for fast (unresolved) transients can be computed by averaging $\bar{n}(x, y)$ over the characteristic diffusion time for the photons.

V. Summary

• We have developed a new technique for independently determining the self-consistent variation of the electron temperature in Comptonizing plasmas.

• The method assumes that $T_e = T_{\rm IC}$ at all times, and utilizes only the information contained in the energy moments of the initial spectrum.

• The sequence of continued fractions obtained in the case of a bremsstrahlung initial spectrum demonstrates uniform convergence to a unique limit function $\theta(y)$, which is the correct temperature variation.

• **The technique essentially decouples the integral and differential parts of the original integrodifferential Kompaneets equation, thereby eliminating the need to use a conventional predictor-corrector algorithm.**

• **The method can be generalized to treat other integrodifferential equations of special interest to the Navy.**

References

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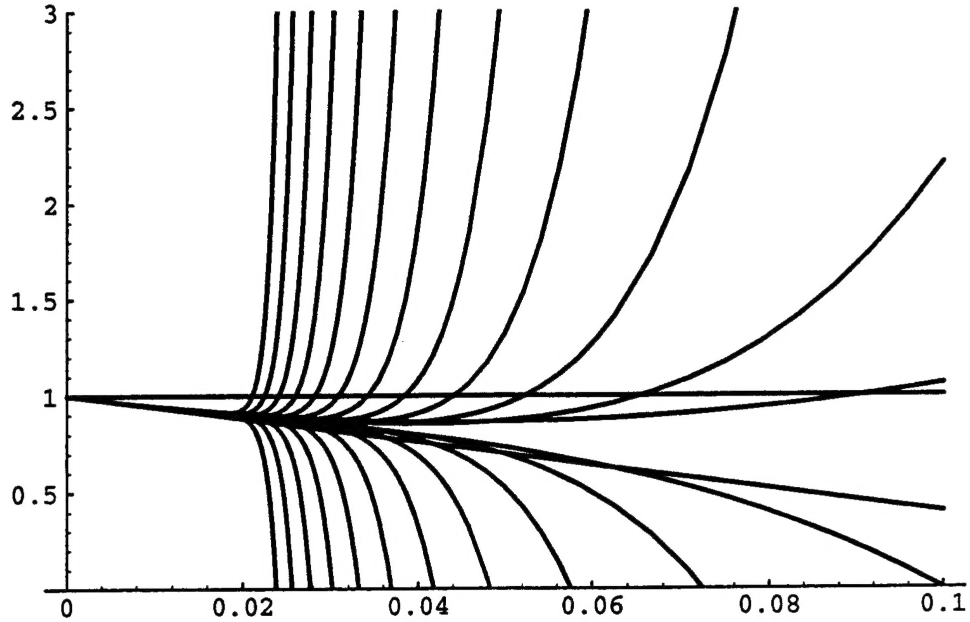

Fig. 1. -- Sequence of Taylor series approximations of order 0 through 24 for the temperature function theta[y] for a bremsstrahlung initial spectrum.

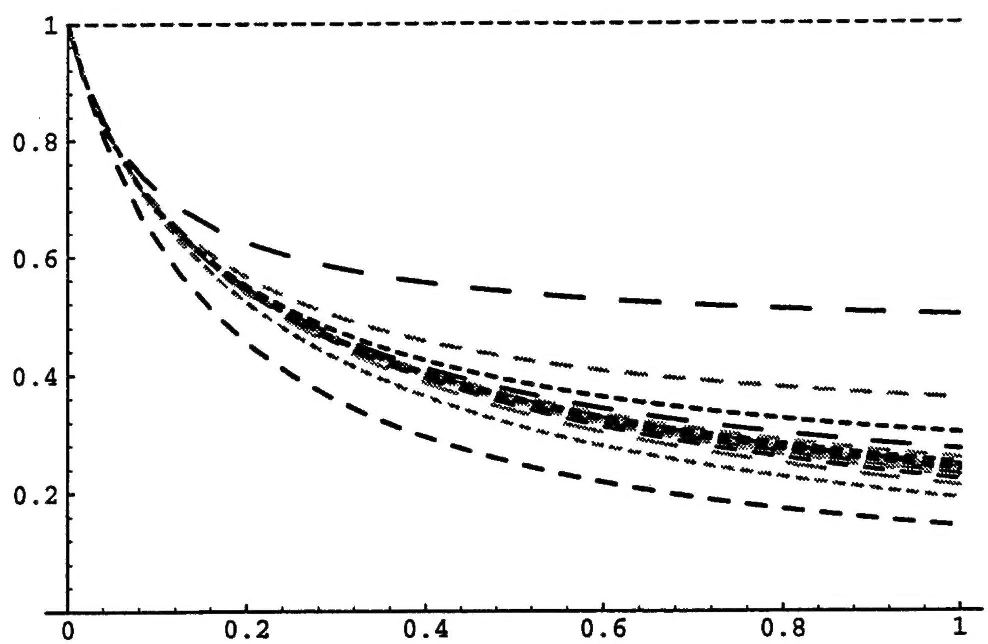

Fig. 2. -- Sequence of continued-fraction approximations of order 0 through 24 for the temperature function theta[y] for a bremsstrahlung initial spectrum.